Linear Algebra 1

10/10/2025

You are **NOT** allowed to use any type of calculators.

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1 Linear equations (10 + 15 = 25 pts)

Three people play a game in which there are always two winners and one looser. They have the understanding that the loser gives each winner an amount equal to what the winner already has. After three games, each has lost just once and each has €24.

(a) Let x_k be the amount of money Player k begins with. Assume that Player 1 has lost the first game, Player 2 the second game, and Player 3 the third game. Write down three equations in terms of x_1, x_2, and x_3 for the amount of money each has after three games.

(b) Solve the linear equations obtained above to determine x_1, x_2, and x_3.

2 Matrix multiplication (2 + 18 = 20 pts)

The *trace* of a square matrix is the sum of its diagonal entries: For $M \in \mathbb{F}^{n\times n}$,

$$\operatorname{trace}(M) = \sum_{i=1}^{n} [M]_{ii}.$$

Let $A \in \mathbb{F}^{q\times r}$ and $B \in \mathbb{F}^{r\times q}$.

(a) What are the sizes of AB and BA?

(b) Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

3 Matrices and their properties (4 + 4 + 4 + 4 + 4 = 20 pts)

Let M be an 4×4 matrix with the characteristic polynomial $p_M(\lambda) = \lambda(\lambda - 1)(\lambda^2 + 1)$.

(a) Is M nonsingular? Justify your answer.

(b) Determine the eigenvalues of M.

(c) Determine the trace of M.

(d) Determine the determinant of M.

(e) Is M diagonalizable? Justify your answer.

4 Determinants and diagonalization (2 + 8 + 3 + 12 = 25 pts)

A *tridiagonal* matrix is a square matrix that has nonzero elements only on the main diagonal, the first diagonal below the main diagonal, and the first diagonal above the main diagonal. Consider the sequence of tridiagonal matrices $A_n \in \mathbb{R}^{n\times n}$ given by:

$$A_1 = 3,\ A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix},\ A_4 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix},\ A_5 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix}, \ldots$$

Let $d_n := \det(A_n)$.

(a) Verify that $d_1 = 3$ and $d_2 = 7$.

(b) By using cofactor expansion, determine the numbers p and q such that $d_k = pd_{k-1} + qd_{k-2}$ for all $k \geqslant 3$.

(c) Note that $\begin{bmatrix} d_k \\ d_{k-1} \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{k-1} \\ d_{k-2} \end{bmatrix}$ for all $k \geqslant 3$ where p and q are as obtained above. Let $M = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$. Show that $d_k = \begin{bmatrix} 1 & 0 \end{bmatrix} M^{k-2} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix}$ for all $k \geqslant 3$.

(d) Determine a nonsingular matrix X and a diagonal matrix D such that $X^{-1}MX = D$. Determine M^k for all $k \geqslant 0$. Determine d_k for all $k \geqslant 0$.

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10 pts free